Nonresonant limit for sequences of resonant orbits: the case of holomorphic maps

1995 J. Phys. A: Math. Gen. 28 2325

(http://iopscience.iop.org/0305-4470/28/8/023)

J. Phys. A: Math. Gen. 28 (1995) 2325–2334. Printed in the UK

Non-resonant limit for sequences of resonant orbits: the case of holomorphic maps

E Todesco and G Turchetti

Dipartimento di Fisica della Università di Bologna, Italy and INFN Sezione di Bologna, Italy

Received 11 July 1994, in final form 28 November 1994

Abstract. The approximation of a non-resonant orbit with a sequence of resonant orbits is considered for the holomorphic maps of the complex plane. The problem is motivated by Hamiltonian dynamics (Greene's conjecture) and we consider a complexified Hamiltonian map in the region (far from the section of real dynamics) where it can be reduced to a holomorphic map of a single complex variable. For a sequence of maps in normal form with linear resonant frequencies, the limit to a linear map with non-resonant diophantine frequency has a simple interpretation: the flower-like resonant orbits become circles due to the increase of the number of petals and the freezing of radial motion. A similar non-trivial result is proved for small perturbations of the normal forms by investigating the behaviour of the conjugation functions.

1. Introduction

The resonant structures for Hamiltonian systems have been intensively investigated: they are the bridge between the ordered quasi-periodic orbits (i.e. KAM tori) and chaotic motion. However, a complete theory is still lacking. The perturbative approach based on Birkhoff normal forms gives rise to an asymptotic series which has a truncation that conveniently describes the geometry of resonant families [1].

Phenomenological analysis has been carried out concerning the way that a family of resonant orbits approaches a non-resonant orbit. This limit is of great importance since it allows one to detect the critical value of a KAM torus. According to the well known Greene conjecture [2], a KAM torus can be approximated by a family of resonant orbits (chains of islands) of increasing period; moreover, before the break up any resonant orbit has an equal number of elliptic and hyperbolic fixed points, whilst after the break up they are all hyperbolic, and the torus becomes an Aubry–Mather Cantor set. This conjecture provides a very accurate computational tool to determine the critical values of invariant curves for area-preserving maps. The statements described above have not been proved, and even the possibility of approximating a KAM torus using a sequence of resonant orbits before the break up has not been demonstrated.

In view of the difficulty of this problem we consider a related problem which can be investigated using the Borel transform techniques. After describing the link which exists between a complexified area-preserving map and a holomorphic map of a single complex variable [3, 4], we show how the invariant curves of a map with a non-resonant linear frequency (Siegel problem [5]) can be approximated by the orbits of a sequence of maps with resonant linear frequencies [6].

0305-4470/95/082325+10$19.50

The corresponding normal forms illustrate the nature of the limit in the 'integrable' case†: the sequences of flower-like orbits of the resonant normal forms converge to the circular orbits of the non-resonant map via the freezing of the radial motion. A similar picture holds in the 'non-integrable' case where we consider a sequence of small perturbations of the resonant normal forms which has a limit that is a linear non-resonant map with a small perturbation. A rigorous proof of the convergence of non-resonant orbits to resonant orbits is based on the convergence of the sequence of the conjugation functions that transform the 'non-integrable' maps into their normal forms. This result has been proved at the first-order in the small perturbation parameter; in this simpler case the conjugation functions satisfy a homological equation rather than a nonlinear functional equation. The conjugation function and its corresponding limit are considered here at the level of the formal series, while the complete proof is given in [7].

The plan of this paper is as follows. In section 2 we give the connection between the Hamiltonian and the holomorphic models. In section 3 we analyse the limit in the 'integrable' case and discuss the change in the geometry of the orbits. In section 4 the statement for the 'non-integrable' case is given and a sketch of the proof is outlined.

2. Hamiltonian models and holomorphic mappings

The complexification of a Hamiltonian map has been considered for the standard map where, for an invariant curve with non-resonant frequency, the images of the analyticity strips in the angle θ were shown to be domains with fractal boundaries [3, 4]. For the vanishing perturbation $\epsilon \to 0$ the width of the strips tends to ∞; the width decreases to zero at the critical break-up value of the invariant curve. For $\epsilon \to 0$ and in the neighbourhood of the analyticity boundaries the map can be approximated in the $z = e^{i\theta}$ plane [8] with a holomorphic map of z.

2.1. Non-resonant case

In order to extend our work to the resonant case we consider complexified area-preserving maps in the neighbourhood of an elliptic fixed point. The limit is easily understood if we consider the non-resonant normal form of the map, namely an integrable mapping, which reads

$$U : \begin{cases} \zeta' = e^{i\Omega(\zeta\eta)}\zeta \\ \eta' = e^{-i\Omega(\zeta\eta)}\eta \end{cases} \qquad \zeta, \eta \in \mathbb{C} \tag{2.1}$$

where Ω is a real analytic function. The real area-preserving map is recovered by setting $\eta = \zeta^*$ (where the * denotes the complex conjugate): one obtains an amplitude-dependent rotation. One can choose ζ and $\rho = \zeta\eta$ (or equivalently (η, ρ)) as independent variables and the map (2.1) reads

$$U : \begin{cases} \zeta' = e^{i\Omega(\rho)}\zeta \\ \rho' = \rho \end{cases} \qquad \zeta, \rho \in \mathbb{C}. \tag{2.2}$$

Fixing ρ we obtain a map of the complex ζ plane with constant frequency.

† We will use the word 'integrable' and 'non-integrable' for holomorphic maps to denote the normal form and perturbations of the normal forms respectively; this terminology is borrowed from Hamiltonian dynamics.

Similarly, a generic polynomial map $\boldsymbol{F} = (f(z,w), g(z,w))$ (where $z, w \in \mathbb{C}^2$) can be expanded in a Laurent series of z with coefficients depending on zw:

$$\begin{cases} z' = f(z,w) = zf_0(zw) + z\sum_{k=1}^{N}(z^k f_k(zw) + z^{-k} f_{-k}(zw)) \\ w' = g(z,w) = wg_0(zw) + w\sum_{k=1}^{N}(w^k g_k(zw) + w^{-k} g_{-k}(zw)) \end{cases} \tag{2.3}$$

where $f_0 = \mathrm{e}^{\mathrm{i}\omega_0} + \mathrm{O}(zw)$, $f_k = \mathrm{O}(1)$, $f_{-1} = \mathrm{O}(zw)$ and $f_{-k} = \mathrm{O}((zw)^{k-1})$. Letting, in this case, $z = \sqrt{r}\,\mathrm{e}^{\mathrm{i}\vartheta}$ and $w = \sqrt{r}\,\mathrm{e}^{-\mathrm{i}\vartheta}$ we see that for the real section $w = z^*$ of the map f_k, the Fourier components are f_{-k}. We can transform the map into a normal form plus a remainder term with a polynomial transformation of order N:

$$\boldsymbol{F} \circ \Phi = \Phi \circ (\boldsymbol{U} + \boldsymbol{E}) \tag{2.4}$$

where $\boldsymbol{E} = \mathrm{O}(|\zeta\eta|^{N/2})$ and the conjugation function Φ, which maps the normal coordinates ζ and η to the initial coordinates z and w, has an expansion similar to (2.3).

Letting (ρ, θ) be defined by $\zeta = \sqrt{\rho}\,\mathrm{e}^{\mathrm{i}\theta}$ and $\eta = \sqrt{\rho}\,\mathrm{e}^{-\mathrm{i}\theta}$, we consider the following two limits:

(i) $\rho \to 0$ and $\mathrm{Im}\,\theta \to -\infty$ keeping ζ finite, which implies $\eta \to 0$

(ii) $\rho \to 0$ and $\mathrm{Im}\,\theta \to +\infty$ keeping η finite, which implies $\zeta \to 0$.

The conjugating function Φ has a Laurent expansion analogous to $\boldsymbol{F}$ (see equation (2.3)), but it can be reduced to a Taylor series in ζ for the first component in limit (i), or to a Taylor series in η for the second component in limit (ii). Indeed, in the first case one has $\zeta^{k+1}\Phi_k(\zeta\eta) = \mathrm{O}(\zeta^{k+1})$ are finite, whereas $\zeta^{-k+1}\Phi_{-k}(\zeta\eta) = \mathrm{O}(\eta^{k-1})$ vanish. For the initial map $\boldsymbol{F}$, limit (i) corresponds to constraining zw to a small value and keeping z finite; its first component f then becomes a Taylor series in z. The function Φ is then given by a Taylor series in ζ, and conjugates $\boldsymbol{F}$ with the linear map $\zeta' = \exp(\mathrm{i}\Omega(\rho))\zeta$.

The analysis can be made rigorous by applying the KAM procedure [9] to the map $\boldsymbol{U}+\boldsymbol{E}$ for any value of ρ such that $\Omega(\rho)$ is diophantine. Then, for very large negative or positive imaginary parts of the angle $\boldsymbol{U} + \boldsymbol{E}$, one defines two analytic maps of $\mathbb{C}$ which can be conjugated with their linear parts.

2.2. *Resonant case*

When the linear frequency of the map ω_0 is close to a resonant value $2\pi p/q$ (p and q integers with no common divisor) then the resonant normal form is given by a map $\boldsymbol{U}$ which commutes with the discrete group of rotations of an angle $2\pi/q$:

$$\boldsymbol{U} : \begin{cases} \zeta' = \mathrm{e}^{\mathrm{i}\Omega(\zeta\eta)}\zeta + \zeta\sum_{\ell=1}^{N/q}(\zeta^{\ell q} A_{\ell q}(\zeta\eta) + \zeta^{-\ell q} A_{-\ell q}(\zeta\eta)) \\ \eta' = \mathrm{e}^{\mathrm{i}\Omega(\zeta\eta)}\eta + \eta\sum_{\ell=1}^{N/q}(\eta^{\ell q} B_{\ell q}(\zeta\eta) + \eta^{-\ell q} B_{-\ell q}(\zeta\eta)) \end{cases} \tag{2.5}$$

where $A_{\ell q} = \mathrm{O}(1)$ and $A_{-\ell q} = \mathrm{O}((\zeta\eta)^{\ell q-1})$. If $\omega_0 = \Omega(0) = 2\pi p/q + \epsilon$ and $\Omega(\rho) = 2\pi p/q$ has a real solution $\overline{\rho}$, then on the plane $\zeta = \eta^*$ where a real area-preserving map is recovered, one has a chain of q islands as suggested by the Birkhoff theorem. The interpolating Hamiltonian, which has a flow that agrees at integer values of time $t = n$ with the nth iterate of the normal form U, has a pendulum-like structure:

$$H(\zeta,\eta) = h(\zeta\eta) + \sum_{\ell=1}^{[N/q]} (\zeta^{\ell q} C_{\ell q}(\zeta\eta) + \zeta^{-\ell q} C_{-\ell q}(\zeta\eta))$$
$$h(\zeta\eta) = \mathrm{i}\epsilon\zeta\eta + h_4(\zeta\eta)^2 + \dots \tag{2.6}$$

where the C_k are related to the A_k, B_k, and $C_k = \mathrm{O}(1)$, $C_{-k} = \mathrm{O}((\zeta\eta)^k)$. At the lowest approximation

$$H = h(\zeta\eta) + \zeta^q C_q(0) + \eta^q C_{-q}(0) \tag{2.7}$$

and in the plane $\zeta = \eta^*$, one has q symmetric islands of area $\sim \epsilon^{q/4}$ which have centres that are a distance $\overline{\rho} \sim \epsilon$ from the origin.

In the exactly resonant case $\lambda_q = \mathrm{e}^{2\pi\mathrm{i}p/q}$, if we constrain $\rho = \zeta\eta$ to a very small value then we obtain from (2.5) a map of $\mathbb{C}$ in the form of a Taylor series in ζ which has a linear part $\lambda_q\zeta$. The corresponding resonant normal form in $\mathbb{C}$ is a map which commutes with the linear part and can be stated as

$$\zeta' = \frac{\lambda_q\zeta}{(1+\zeta^q)^{1/q}} \qquad \lambda_q = \mathrm{e}^{\mathrm{i}2\pi p/q}. \tag{2.8}$$

In the most general case q is replaced by kq in the denominator of (2.8), but for simplicity we consider $k = 1$.

Let us consider the orbits of an area-preserving map in the neighbourhood of a chain of q islands and, more precisely, only those orbits which join the q hyperbolic fixed points (i.e. the separatrices); if we let all the hyperbolic fixed points collapse on the origin, we obtain a mapping on the separatrices which have dynamics and phase space analogous to the resonant normal form of period q of the holomorphic mapping (2.8). In the following section we will illustrate the dynamics of this map, showing how in the limit $q \to \infty$ it is possible to recover a circular motion, which in the Hamiltonian case is analogous to approximating a KAM curve with a sequence of separatrices issued from hyperbolic fixed points of increasing order.

3. The 'integrable' case and the geometry of the orbits

We consider the maps of $\mathbb{C}$ and describe how a sequence of normal forms with resonant frequencies can approximate a linear map with a non-resonant diophantine frequency; this corresponds to the integrable case of Hamiltonian dynamics since the maps exhibit explicit iterations and invariants of motion. The normal form [10, 11] given by the standard shift (2.8) commutes with the symmetry group $\zeta' = \lambda_q\zeta$ generated by the linear part of the map, where λ_q is the qth root of unity. The map has a trivial iteration

$$\zeta(t) = \frac{\lambda_q^t\zeta}{(1+t\zeta^q)^{1/q}} \tag{3.1}$$

for integer t which can be extended to $t \in \mathbb{R}$ in order to build an interpolating flow, and has explicit invariant $I(\zeta) = \exp(2\pi\mathrm{i}/\zeta^q)$. The dynamics of the map follows the well known

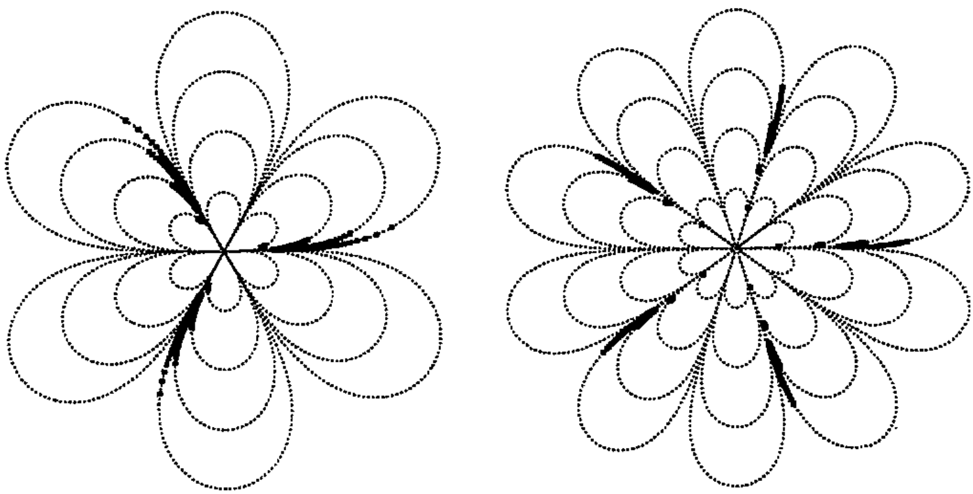

Figure 1. Phase portrait of the interpolating flow (solid line) and of the first 1000 iterates (dots) relative to four different initial conditions of the standard shift with $q = 3$ (left-hand side) and $q = 5$ (right-hand side).

flower structure [6]: in figure 1 the orbits of the interpolating flow (solid line) and the first 1000 iterates of the map (dots) are shown in the case $k = 1$, $q = 3$ (left-hand side) and $k = 1$, $q = 5$ (right-hand side).

The normal form dynamics can be decomposed into two components: the first component, responsible for the motion along the petal, is a map tangential to the identity with interpolating flow

$$\zeta(t) \to \frac{\zeta}{(1 + t\zeta^q)^{1/q}} \tag{3.2}$$

and the second component, responsible for the angular motion, is determined by the linear map with interpolation $\lambda_q^t \zeta$. The linear map $\zeta' = \lambda_q \zeta$ determines a jump from one petal to another by skipping $p - 1$ petals. When $p, q \to \infty$ and $\lambda_q \to \lambda = \exp(i\omega)$, the normal form tends uniformly in $|\zeta| \leqslant R < 1$ to the linear part, independently of the arithmetic properties of the frequency ω. From the geometrical point of view the transformation of the petals into closed circles is caused by the freezing of the radial motion on the petals. In fact, if we compute the radial velocity of the interpolating flow (3.2) we find

$$\dot{\zeta}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\zeta}{(1 + t\zeta^{kq})^{1/kq}}\right) = -\frac{[\zeta(t)]^{q+1}}{q} \to 0. \tag{3.3}$$

In the limit $q \to \infty$ the motion along the petal is frozen and only the angular motion from one petal to another due to the linear map remains active. When q is increased the petals become thinner and thinner; in the limit $q \to \infty$ they disappear, giving rise to periodic or dense orbits on the circle if $\omega/(2\pi)$ is rational or irrational.

This is shown in figures 1 and 2 which plot the orbits of the interpolating flow (solid lines) for increasing values of $q = 3$, 5, 8 and 13 and for 1000 iterates of the map. In the case of $q = 8$ and 13 one already observes the freezing of the radial motion along the petals (see figure 2).

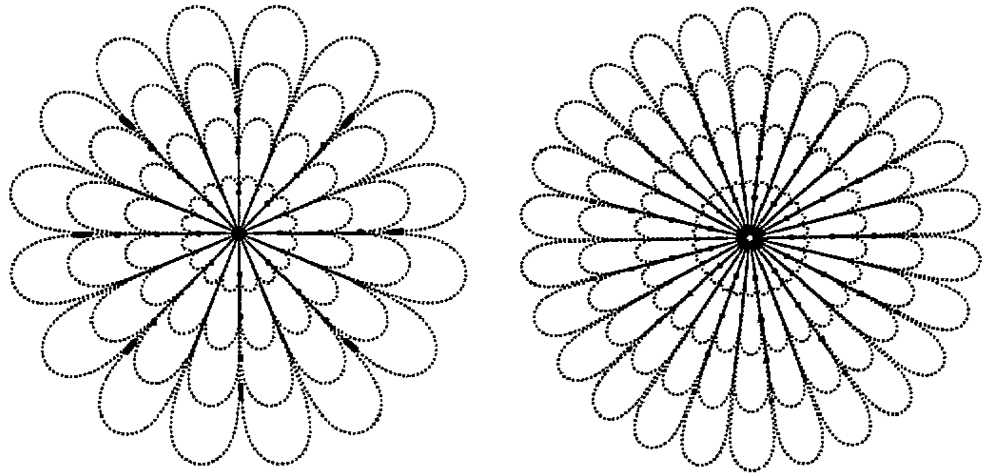

Figure 2. Phase portrait of the interpolating flow (solid line) and of the first 1000 iterates (dots) relative to four different initial conditions of the standard shift with $q = 8$ (left-hand side) and $q = 13$ (right-hand side).

4. The 'non-integrable' case

We now consider a sequence of mappings which are small perturbations of the normal form (2.8) corresponding to the non-integrable case of Hamiltonian dynamics:

$$F_q(z) = u_q(z) + \epsilon f(z) \qquad u_q(z) = \frac{\lambda_q z}{(1+z^q)^{1/q}} \qquad f(z) = \sum_{n=2}^{\infty} f_n z^n \tag{4.1}$$

where $f(z)$ is taken independent of q and analytic on the disc $D_R = \{|z| \leqslant R\}$ with $R < 1$. Contrary to the previous case, we have to restrict the frequency of the limit mapping to be sure that invariant closed curves exist in the neighbourhood of the origin, in accordance with the theory developed by Siegel [5], Brjuno [12] and Yoccoz [13]. Therefore we assume that the eigenvalue $\lambda_q = \exp(2\pi i p/q)$ tends to $\lambda = \exp(i\omega)$, and ω satisfies the Brjuno condition for the homological equation:

$$\frac{\log q_{r+1}}{q_r} < +\infty \qquad \forall r \in \mathbb{N} \tag{4.2}$$

where q_r are the denominators of the approximations of ω given by the continued fraction [5]. If ω is a diophantine number, condition (4.2) is always satisfied.

When $q \to \infty$, the resonant map F_q tends uniformly in D_R to the mapping

$$F(z) = \lambda z + \epsilon f(z). \tag{4.3}$$

The orbits of $F(z)$ in the neighbourhood of the origin are deformed closed circles and one can build a convergent transformation Ψ which conjugates the map to its normal form, i.e. the linear part $\zeta' = \lambda\zeta$. If one expands the transformation in a power series of ϵ, $\zeta = \Psi(z) = z + \epsilon\psi(z) + O(\epsilon^2)$, the first-order term satisfies the homological equation $\psi(\lambda z) - \lambda\psi(z) = -f(z)$ which has the solution

$$\psi(z) = \sum_{n=2}^{+\infty} \frac{f_n}{\lambda - \lambda^n} z^n. \tag{4.4}$$

Using the Brjuno condition one can prove that ψ is analytic in a neighbourhood of the origin.

In order to analyse the dynamics of the small perturbation F_q of the resonant normal form, we perform a conjugation with the integrable map and discuss the analytic properties of the conjugating function for the homological equation. In order to simplify the analysis, we conjugate the resonant map F_q to a preliminary normal form at first-order in ϵ defined as

$$U_q(\zeta) = u_q(\zeta) + \epsilon u_q^{(1)}(\zeta) \qquad u_q^{(1)}(\zeta) = \sum_{j=1}^{\infty} u_{qj+1}\, \zeta^{qj+1} \tag{4.5}$$

through the transformation

$$\zeta = \Psi_q(z) = z + \epsilon \psi_q(z) + \mathrm{O}(\epsilon^2) \tag{4.6}$$

which contains only non-resonant terms

$$\psi_q(z) = \sum_{j\neq kq+1, j\geqslant 2}^{\infty} \psi_{q,j} z^j \equiv \sum_{j}^{*} \psi_{q,j} z^j. \tag{4.7}$$

At first-order in ϵ, the functional equation $\Psi_q \circ F_q = U_q \circ \Psi_q$ reads explicitly as

$$\psi_q(u_q(z)) + f(z) = u_q'(z)\psi_q(z) + u_q^{(1)}(z) \tag{4.8}$$

(where $u_q'(z)$ is the derivative of $u_q(z)$ with respect to z) and can be solved by projecting into the subspace of normal forms† and its complement. Letting Π be the projector into the space of normal forms, we obtain

$$\begin{aligned} &u_q^{(1)} = \Pi f \\ &\psi_q(u_q(z)) - u_q'(z)\psi(z) + (1-\Pi) f(z) = 0. \end{aligned} \tag{4.9}$$

The last equation can be solved explicitly if we replace u_q' with λ_q. To achieve this we add a remainder term to the preliminary normal form, that is we choose

$$U_q(\zeta) = u_q(\zeta) + \epsilon u_q^{(1)}(\zeta) + \epsilon r_q(\zeta) \tag{4.10}$$

where $r_q(\zeta) = \mathrm{O}(\zeta^{q+2})$ is not in normal form. Then the remainder term appears in the r.h.s. of equation (4.8) which can be written as

$$\psi_q(u_q(z)) - \lambda_q \psi_q(z) + f(z) = u_q^{(1)}(z) - [\lambda_q - u_q'(z)]\psi_q(z) + r_q(z) \tag{4.11}$$

and is solved according to

$$\begin{cases} \psi_q(u_q(z)) - \lambda_q \psi_q(z) = (1-\Pi) f(z) \equiv \sum^{*} f_j z^j \\ u_q^{(1)}(z) = \Pi f(z) \qquad r_q(z) = [\lambda_q - u_q'(z)]\psi_q(z). \end{cases} \tag{4.12}$$

† The subspace of normal forms is the set of monomials which are in normal form.

In the following, we will prove that the formal solution of the functional equation (4.12) for ψ_q is given by

$$\psi_q(z) = \frac{1}{\lambda_q}\sum_j^* f_j z^j \sum_{k=0}^{\infty} \lambda_q^{k(j-1)} \frac{1}{(1+kz^q)^{j/q}} = \sum_{j=2}^{q} \frac{f_j}{\lambda_q - \lambda_q^j} z^j + \mathrm{O}(z^{q+2}) \tag{4.13}$$

and, therefore, in the limit $q \to \infty$ and $\lambda_q \to \lambda$, one formally recovers the conjugation function of the homological Siegel problem (4.4). Similarly, one can verify that the preliminary normal form $u_q^{(1)}(\zeta)$ and the remainder $r_q(\zeta)$ formally vanish in the limit $q \to \infty$; as a consequence the preliminary normal form plus the remainder formally converge to the normal form.

In order to prove (4.13), we introduce the mapping $w = z^{-q}$ of $\mathbb{C}$ to $\mathbb{C}^q$, and we define

$$\overline{\psi}(w) \equiv \psi(w^{-1/q}) = \psi(z) \qquad \overline{f}(w) \equiv (1-\Pi) f(w^{-1/q}) = (1-\Pi) f(z) \tag{4.14}$$

which are given by the series expansions

$$\overline{\psi}_q(w) = \sum_{n=1}^{\infty} \psi_{q,n} w^{-n/q} \qquad \overline{f}(w) = \sum^* f_n w^{-n/q}. \tag{4.15}$$

The first functional equation of (4.12) then reads

$$\overline{\psi}\left(\mathrm{e}^{-2\pi\mathrm{i}p}(1+w)\right) - \lambda_q \overline{\psi}(w) + \overline{f}(w) = 0. \tag{4.16}$$

A solution can be found introducing the Borel transforms $\psi_{q\mathrm{B}}(t)$ and $f_{\mathrm{B}}(t)$ of $\overline{\psi}(w)$ and $\overline{f}(w)$ respectively:

$$\overline{\psi}_q(w) = w^{-1}\int_0^{\infty} \mathrm{e}^{-\tau} \psi_{q\mathrm{B}}\left(\frac{\tau}{w}\right) \mathrm{d}\tau = \int_0^{w^{-1}\infty} \mathrm{e}^{-tw} \psi_{q\mathrm{B}}(t)\, \mathrm{d}t \tag{4.17}$$

where in the second integral the integration path on $t = \tau/w$ is a straight line emerging from the origin with the phase of w^{-1}, and $\psi_{q\mathrm{B}}$ is the Borel transform defined by the analytic continuation of the series

$$\psi_{q\mathrm{B}}(t) = \sum_{n=1}^{\infty} \psi_{q,n} \frac{t^{n/q-1}}{\left(\frac{n}{q}-1\right)!}. \tag{4.18}$$

In the second integral of (4.17) the integration path can be changed to straight lines with a phase different from that of w^{-1} to perform the analytic continuation in w [14].

Using the relation $g_{\mathrm{B}}(t) = \mathrm{e}^{-t} a f_{\mathrm{B}}(at)$, where $g(w) = f(a^{-1}(w+1))$, the Borel transform of (4.16) reads

$$\mathrm{e}^{-t}\overline{\psi}_{q\mathrm{B}}(\mathrm{e}^{2\pi\mathrm{i}p}t) - \lambda_q \overline{\psi}_{q\mathrm{B}}(t) + \overline{f}_{\mathrm{B}}(t) = 0. \tag{4.19}$$

Replacing $t \to \mathrm{e}^{2\pi\mathrm{i}p}t$ in (4.18) and multiplying by $\mathrm{e}^{-t}\lambda_q^{-1}$ one obtains a new equation. Repeating this procedure on the new equation q times one obtains q equations; taking the sum, all the terms with $\overline{\psi}_{q\mathrm{B}}(\mathrm{e}^{2\pi\mathrm{i}kp}t)$ with $k = 1, \ldots, q-1$ cancel and one obtains

$$\overline{\psi}_{q\mathrm{B}}(t) = \frac{1}{\lambda_q(1-\mathrm{e}^{-qt})} \sum_{k=0}^{q-1} \lambda_q^{-k} \mathrm{e}^{-kt} \overline{f}_{\mathrm{B}}(\mathrm{e}^{2\pi\mathrm{i}kp}t). \tag{4.20}$$

Replacing the expression of $\overline{f}_B(t)$ in (4.20) we obtain

$$\begin{aligned}\overline{\psi}_{qB}(t) &= \frac{1}{\lambda_q(1-e^{-qt})}\sum_{k=0}^{q-1}\frac{e^{-kt}}{\lambda_q^k}\sum_{\ell}^{*}\frac{f_\ell t^{\ell/q-1}}{\left(\frac{\ell}{q}-1\right)!}e^{2\pi ikp(\ell/q-1)} \\ &= \sum_{\ell}^{*}\frac{f_\ell}{\left(\frac{\ell}{q}-1\right)!}\frac{t^{\ell/q-1}}{\lambda_q-\lambda_q^\ell e^{-t}}.\end{aligned} \tag{4.21}$$

Taking the Laplace transform after some algebraic manipulation, result (4.12) is obtained.

It can be proved that both U_q and ψ_q are given by a collection of $2q$ functions analytic on sectors of aperture smaller than $2\pi/q$. In the limit $q \to \infty$ and $\lambda_q \to \lambda$, one can easily verify that U_q formally converges to $U = \lambda\zeta$, and ψ_q formally tends to the solution of the homological equation (4.4). Moreover, using the subsequence of q_r, p_r given by the continued fraction expansion of $\omega/2\pi$, one can prove that the convergence is not only formal but that analyticity in a close neighbourhood of the origin is recovered in the limit. More precisely, the following result holds.

Proposition. If ω satisfies the Brjuno condition and the subsequence q_r, p_r approximates ω according to the algorithm of the continued fraction, then for $r \to \infty$, ψ_{q_r} uniformly converges to ψ on a neighbourhood of the origin.

The proof is based on the Cauchy estimates of the remainders of the series. Using the Borel and the Laplace transform, one can prove that the formal series defining the conjugation function can be re-summed to $2q$ functions, analytic in sectors of aperture smaller than $2\pi/q$. The Borel transform has poles with residues different to zero that are responsible for the sectorial analyticity, and, therefore, for the divergence of the Taylor series. In the limit $q \to \infty$ a function analytic in a disc is recovered since, provided that $\omega_q \to \omega$ where $\omega/(2\pi)$ is a Brjuno number [12], these residues vanish. A Cauchy estimate of these $2q$ functions show that they all uniformly converge to the same function, analytic in a disc.

Acknowledgments

We wish to acknowledge Professor J P Ramis and Dr F Fauvet for helpful suggestions and stimulating remarks; we also want to thank Professors J C Canille Martins and C Camacho for helpful discussions, and IMPA for their hospitality during the summer of 1992 when this work was started.

References

[1] Bazzani A, Giovannozzi M, Servizi G, Todesco E and Turchetti G 1993 *Physica* **64D** 66
[2] Greene J M 1979 *J. Math. Phys.* **20** 1183
[3] Percival I C and Greene J M 1981 *Physica* **3D** 530
[4] Percival I C 1982 *Physica* **6D** 67
[5] Siegel L 1942 *Ann. Math.* **43** 607
[6] Camacho C 1978 *Astérisque* **59–60** 83
[7] Todesco E and Turchetti G 1995 Approximation of linearizable mappings of (C, 0) through resonant diffeomorphisms *Ann. Inst. Fourier* submitted

[8] Billi L, Malavasi M and Turchetti G 1995 *Integrability and Chaotic Behaviour* ed J Seimenis (New York and London: Plenum) pp 35–42
[9] Bazzani A and Turchetti G 1992 *J. Phys. A: Math. Gen.* **25** 427
[10] Ecalle J 1982 *Les fonctions Résurgentes et Leurs Applications* (Orsay: Publication Math. d'Orsay)
[11] Martinet J and Ramis J P 1983 *Ann. scient. Éc. Norm. Sup.* **16** 571
[12] Brjuno A D 1990 *IHES* M 36
[13] Yoccoz J C 1988 *C. R. Acad. Sci. Paris I* **306** 55
[14] Eckmann J P and Witter P 1985 *Computer Methods and Borel Summability Applied to Feigenbaum's Equation (Lecture Notes in Physics 227)* (Berlin: Springer)